# ON THE SINGULARITIES OF SINGLE-VALUED AND GENERALLY ANALYTIC FUNCTIONS.

By PROF. ALEXANDRE S. CHESSIN, Baltimore, Md.

A single-valued function of one variable $f(z)$ will be called *generally analytic* in a domain $(D)$ if it be analytic everywhere in this domain except at points and lines forming a *discrete multiplicity*.* It is well known that such functions have no singularities in the domain $(D)$ other than poles or essentially singular points. However, the proof of this fundamental proposition is lacking in rigor in most of the existing treatises on the Theory of Functions, and it may therefore be desirable to supply this deficiency. The weak point of these demonstrations consists in admitting without proof a proposition which is by no means evident, namely: *If at a point $a$ the single-valued and generally analytic function $f(z)$ ceased to be analytic without becoming infinite, then the function $(z - a)f(z)$ would be analytic at this point.*† As a matter of fact all we can affirm is that the function $\varphi(z) = (z - a)f(z)$ is finite and continuous at the point $a$. But this is only one of the conditions implied in the definition of an analytic function; nothing shows that the condition

$$\frac{\partial \varphi(z)}{\partial y} = i\,\frac{\partial \varphi(z)}{\partial x} \tag{1}$$

is satisfied at this point *à priori*. We will show in this paper that the condition (1) is really satisfied at the point $a$ and that $f(z)$ cannot cease to be analytic without becoming infinite.

Our proof is based on the following corollary of Green's Theorem: ‡

*Let $X(x, y)$ and $Y(x, y)$ be two functions of the real variables $x$ and $y$, which are finite and continuous throughout a connected domain $(D)$, and which generally* (i. e. with the exception of points and lines forming a discrete multi-

* We will say that a finite or infinite number of points and lines form a discrete multiplicity if they can be included within areas $\sigma_1$, $\sigma_2$, $\sigma_3$, . . . such that $\Sigma_i \sigma_i$ can be made arbitrarily small. In this definition we follow the ideas of Hankel. See for example Harnack's Introduction to the study of the Differential and Integral Calculus (English translation by G. Cathcart), pp. 243, 296.

† See for example *Forsyth's* Theory of Functions, §§ 32–33, pp. 52 and 53; or *Durège's* Elements of the Theory of Functions (English translation from the 4th German edition), p. 127.

‡ This form of Green's theorem where Rieman's notion of integrability is introduced is due to A. Harnack. See, for example, his Introduction to the Study of the Diff. and Int. Calculus, p. 315.

plicity) *satisfy the equation* $\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$; *let further* $(x_0, y_0)$ *and* $(x, y)$ *be any two points within* $(D)$; *then the definite integral*

$$\int_{(x_0, y_0)}^{(x, y)} (Xdx + Ydy) \tag{2}$$

*will be independent of the path of integration, provided the several paths lie entirely within* $(D)$ *and can be brought to coincide with one another by a continuous deformation without crossing any of the boundary lines of* $(D)$.

It is well to remark that the limits of the curvilinear integral (2) may be points at which the equation $\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$ is not satisfied.

Let us now put $f(z) = u + iv$, and let us consider the definite integral $\int_{z_0}^{z} f(z)\,dz$. This integral is equal to the sum of two curvilinear integrals, namely:

$$\int_{z_0}^{z} f(z)\,dz = \int_{(x_0, y_0)}^{(x, y)} (udx - vdy) + i\int_{(x_0, y_0)}^{(x, y)} (vdx + udy)\,.$$

To each one of these curvilinear integrals we may apply the above proposition, and we thus obtain the following :—

*Theorem I. Let* $f(z)$ *be a single-valued function which is finite and continuous throughout a connected domain* $(D)$ *and generally analytic in the same; let also* $z_0$ *and* $z$ *be any two points within* $(D)$; *then the definite integral*

$$\int_{z_0}^{z} f(z)\,dz$$

*will be independent of the path of integration, provided the several paths lie entirely within* $(D)$ *and can be brought to coincide with one another by a continuous deformation without crossing any of the boundary lines of* $(D)$.

In fact the functions $u(x, y)$ and $v(x, y)$ are finite and continuous throughout the domain $(D)$, and, moreover, they *generally* satisfy the equations $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$; $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$, since $f(z)$ is generally analytic in the domain $(D)$ by hypothesis.

Theorem I is a generalization of Cauchy's theorem, which assumes that $f(z)$ is analytic *throughout* the given domain.

We will now show that a single-valued and generally analytic function $f(z)$ cannot be finite and continuous throughout a connected domain unless it be analytic without exception in the same. To this end we will prove the following:—

*Theorem II. Let the single-valued function $f(z)$ be finite and continuous throughout a connected domain $(D)$ and generally analytic in the same. If then we put*

$$F(z) = \int_{z_0}^{z} f(z)\,dz \tag{3}$$

*we shall have $F'(z) = f(z)$ throughout the domain $(D)$.*

In fact let $z + \Delta z$ be any point in the neighborhood of the point $z$ within the domain $(D)$, and let us consider the integral

$$\int_{z}^{z+\Delta z} f(z)\,dz\,. \tag{4}$$

By Theorem I we may take for the path of integration in (4) the broken line from $z$ to $z + \Delta x$, and thence to $z + \Delta z$. This broken line will always lie within $(D)$ provided $\Delta z$ be sufficiently small. Hence

$$\begin{aligned}\int_{z}^{z+\Delta z} f(z)\,dz &= \int_{z}^{z+\Delta x} f(z)\,dz + \int_{z+\Delta x}^{z+\Delta z} f(z)\,dz \\ &= \int_{x}^{x+\Delta x} f(x+iy)\,dx + i\int_{y}^{y+\Delta y} f(x+\Delta x+iy)\,dy\,.\end{aligned}$$

On the other hand, $f(z)$ being by hypothesis continuous throughout the given domain, we have by a well known proposition

$$\int_{x}^{x+\Delta x} f(x+iy)\,dx = \Delta x\,[f(x+iy)+\varepsilon_1] = \Delta x\,[f(z)+\varepsilon_1]$$

$$\int_{y}^{y+\Delta y} f(x+\Delta x+iy)\,dx = \Delta y\,[f(x+\Delta x+iy)+\varepsilon_2] = \Delta y\,[f(z)+\varepsilon_3]$$

where $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$ are arbitrarily small quantities vanishing with $\Delta z$. Hence

$$F(z+\Delta z) - F(z) = \int_{z}^{z+\Delta z} f(z)\,dz = \Delta z\,.\,f(z) + \varepsilon_1\Delta x + i\varepsilon_3\Delta y\,,$$

from which follows

$$\frac{F(z+\Delta z)-F(z)}{\Delta z}=f(z)+\frac{\varepsilon_1\Delta x+i\varepsilon_3\Delta y}{\Delta z}.$$

But

$$\left|\frac{\varepsilon_1\Delta x+i\varepsilon_3\Delta y}{\Delta z}\right|$$

is an arbitrarily small quantity vanishing with $\Delta z$; hence

$$F'(z)=\underset{\Delta z=0}{Lim}\frac{F(z+\Delta z)-F(z)}{\Delta z}=f(z)\,. \qquad \text{Q. E. D.}$$

Theorem II may be stated also as follows: *The definite integral of $f(z)$ considered as function of its upper limit is an analytic function throughout the domain $(D)$.*

COROLLARY. *A function $f(z)$ which is single-valued and generally analytic in a given domain becomes discontinuous at the points at which it ceases to be analytic.*

In fact, if $f(z)$ were continuous at a point $a$ at which it ceases to be analytic, let $(D)$ be the neighborhood of this point. By Theorem II the integral function $F(z)$ would be analytic and $F'(z)=f(z)$ within $(D)$, i. e. in the neighborhood of the point $a$. But we know that the derivative of an analytic function is itself an analytic function in the same domain. Hence $f(z)$ would be analytic at the point $a$, contrary to our assumption.

We are able now to prove that the condition (1) is really satisfied at a point $a$, if at this point $f(z)$ ceased to be analytic without becoming infinite. In fact then the function $\varphi(z)=(z-a)f(z)$ which is single-valued and generally analytic in the same domain as $f(z)$, being moreover continuous in the neighborhood of the point $a$, cannot cease to be analytic at this point by the above Corollary. Hence, the equation (1) will be satisfied at the point $a$. Q. E. D.

This proposition being established it remains only to proceed in the usual way to show that a single-valued and generally analytic function can have no singularities other than poles or essentially singular points. The function $\varphi(z)$ being single-valued and analytic at the point $a$ we can develop it by Taylor's theorem into a series

$$\varphi(z)=(z-a)f(z)=c_0+c_1(z-a)+c_2(z-a)^2+\dots.$$

The function $f(z)$ is by hypothesis finite; therefore $\lim\limits_{z=a} [(z-a)f(z)] = 0$ and in consequence $c_0 = 0$. Hence

$$f(z) = c_1 + c_2(z-a) + \ldots,$$

which shows that $f(z)$ is analytic at the point $a$ contrary to our assumption. We see, therefore, that *a single-valued and generally analytic function $f(z)$ cannot cease to be analytic without becoming at the same time infinite.*

Now, there are two ways in which a function may become infinite at a point $a$: Either $\lim\limits_{z=a} f(z) = \infty$ *uniformly,* i. e. whatever be the path along which the point $z$ tends to the point $a$; or $\lim\limits_{z=a} f(z) = \infty$ *non-uniformly,* i. e. only as $z$ tends to $a$ along certain paths while along other paths the value of $\lim\limits_{z=a} f(z)$ is different. Singular points of the first kind are called poles of $f(z)$; singular points of the second kind are called *essentially singular* points of $f(z)$. Thus the study of the singularities of single-valued and generally analytic functions is reduced to the study of these two types.

NOTE.—This paper was written and sent to the Annals of Mathematics by the author some time last May, *i. e.* before the appearance of Mr. W. F. Osgood's article on "some points in the elements of the theory of functions" in the Bulletin of the Amer. Math. Society (June, 1896). Mr. Osgood gives two very interesting and simple proofs of the proposition discussed in this paper. The demonstration given here is really a modification of Riemann's proof, and is extracted from the author's lectures on the elements of the theory of functions at the Johns Hopkins University.